Sparse spikes deconvolution on thin grids

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Outline

1. The deconvolution problem

2. Discrete regularization

3. The continuous limit

Summary

Deconvolution

Measuring devices have a non sharp impulse response: our observations are **blurred** of a "true ideal scene".

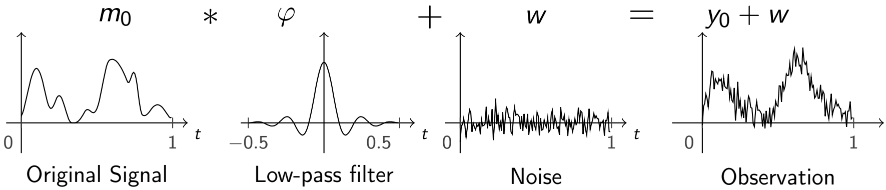
- Geophysics,
- Astronomy,
- Microscopy,
- Spectroscopy,
- ...

Image courtesy of S. Ladjal

Goal: Obtain as much detail as we can from given measurements.

The Deconvolution Problem

- Consider a signal m_0 defined on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (i.e. $[0, 1)$ with periodic boundary condition).
- Perturbation model:

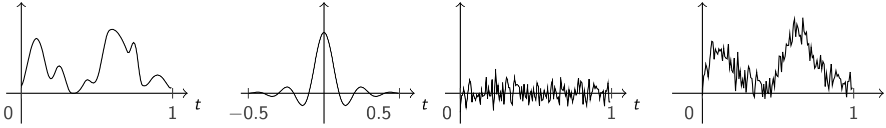


- Goal: recover m_0 from the observation $y_0 + w = \varphi * m_0 + w$ (or simply $y_0 = \varphi * m_0$)

The Deconvolution Problem

- Consider a signal m_0 defined on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (i.e. $[0, 1)$ with periodic boundary condition).
- Perturbation model:

$$m_0 \quad * \quad \varphi \quad + \quad w \quad = \quad y_0 + w$$

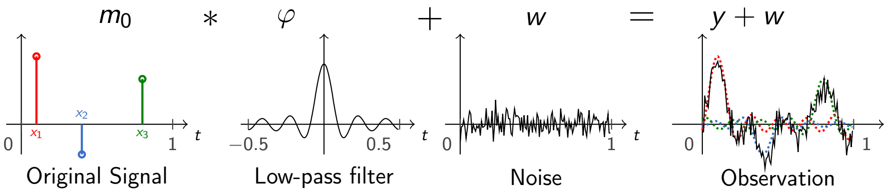


Original Signal | Low-pass filter | Noise | Observation

- Goal: recover m_0 from the observation $y_0 + w = \varphi * m_0 + w$ (or simply $y_0 = \varphi * m_0$)
- Ill-posed problem:
  - the low pass filter might not be invertible ($\hat{\varphi}_n = 0$ for some frequency n)
  - even though, the problem is ill-conditioned ($|\hat{\varphi}_n| \ll |\hat{\varphi}_0|$ for high frequencies n)

The Deconvolution Problem

- Assumption: the signal m_0 is **sparse**



$$m_0 = \sum_{i=1}^{N} a_i \delta_{x_i}, \quad \text{where} \begin{cases} a_i \in \mathbb{R}, \\ x_i \in \mathbb{T}, \\ N \in \mathbb{N} \text{ is small.} \end{cases}$$

so that we observe $y + w = \sum_{i=1}^{N} a_i \varphi(\cdot - x_i) + w$.

- Idea: Look for a **sparse** signal m such that $\varphi * m \approx y_0 + w$ (or y_0).

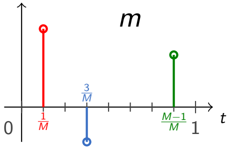
Question

Can we guarantee that the reconstructed signal is close to the original one?

Summary

Discretization

Define a finite grid $\mathcal{G} = \{\frac{i}{M};\ 0 \leqslant i \leqslant M-1\} \subset \mathbb{T}$, and consider signals of the form $m = \sum_{i=0}^{M-1} a_i \delta_{\frac{i}{M}}$.



Candidate Signal

- Write

$$\varphi * m = \sum_{i=0}^{M-1} a_i \varphi(\cdot - \frac{i}{M})$$

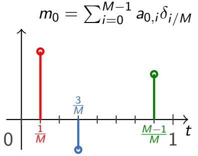
$$= \underbrace{\left(\begin{array}{c|c|c|c} \varphi & \varphi(\cdot - \frac{1}{M}) & \ldots & \varphi(\cdot - \frac{M-1}{M}) \end{array} \right)}_{\Phi} \underbrace{\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{M-1} \end{pmatrix}}_{a}.$$

- Equivalent paradigm: Look for a **sparse** vector $a \in \mathbb{R}^M$ such that $\Phi a \approx y_0$ (or $\Phi a \approx y_0 + w$).

Discrete ℓ^1 regularization

Define

$$\|m\|_{\ell^1(\mathcal{G})} = \begin{cases} \sum_{i=0}^{M-1} |a_i| & \text{if } m = \sum_{i=0}^{M-1} a_i \delta_{i/M}, \\ +\infty & \text{otherwise.} \end{cases}$$



- **Basis Pursuit** [Chen & Donoho (94)]

$$\inf_{m \in \mathcal{M}(\mathbb{T})} \|m\|_{\ell^1(\mathcal{G})} \text{ such that } \Phi m = y_0 \qquad (\mathcal{P}_0^M(y_0))$$

- **LASSO** [Tibshirani (96)] or **Basis Pursuit Denoising** [Chen et al. (99)]

$$\inf_{m \in \mathcal{M}(\mathbb{T})} \lambda \|m\|_{\ell^1(\mathcal{G})} + \frac{1}{2}\|\Phi m - (y_0 + w)\|_2^2 \qquad (\mathcal{P}_\lambda^M(y_0 + w))$$

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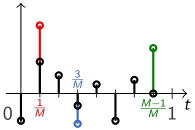

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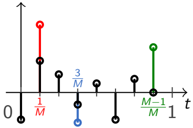
ℓ^2-robustness (Grasmair et al. (2011))

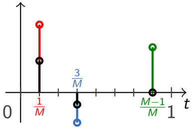
If $m_0 = \sum_i a_{0,i}\delta_{i/M}$ is the unique solution to $\mathcal{P}_0^M(y_0)$, and $m_\lambda = \sum_i a_{\lambda,i}$ is a solution to $\mathcal{P}_\lambda^M(y_0 + w)$, then $\|a_\lambda - a_0\|_2 = \mathcal{O}(\|w\|_2)$ for $\lambda = C\|w\|_2$.

Robustness of the support (discrete problem)

$m_0 = \sum_{i=0}^{M-1} a_{0,i}\delta_{i/M}$



No support recovery



Support recovery

Can one guarantee that $\operatorname{Supp} m_\lambda = \operatorname{Supp} m_0$?

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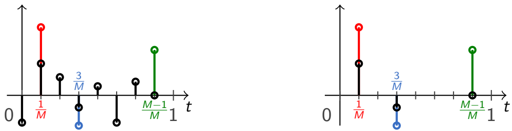


Can one guarantee that $\operatorname{Supp} m_\lambda = \operatorname{Supp} m_0$?

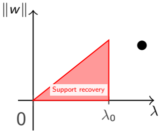
- Sufficient conditions for $\operatorname{Supp} m_\lambda \subseteq \operatorname{Supp} m_0$:
  - **Exact Recovery Principle (ERC)** [Tropp (06)]
  - **Weak Exact Recovery Principle (W-ERC)** [Dossal & Mallat (05)]
- Almost necessary and sufficient $\operatorname{Supp} m_\lambda = \operatorname{Supp} m_0$
  - **Fuchs criterion** [Fuchs (04)]

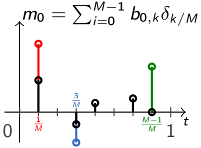
  See also [Vaiter et al. (14)] for more general regularizers, [Liang et al. (15)] for implications on the convergence rates of optimization methods.

Fuchs theorem

For $m_0 = \sum_{i=1}^{M} a_{0,i}\delta_{x_{\mathbf{0},i}}$, define

$\eta_F =$





Theorem (Fuchs (04))

Assume that $\Phi_{x_{\mathbf{0}}} \stackrel{\text{def.}}{=} (\varphi(\cdot - x_{0,1}), \ldots \varphi(\cdot - x_{0,N}))$ *has full rank.*
If $|\eta_F(\frac{k}{M})| < 1$ *for all* k *such that* $\frac{k}{M} \notin \{x_{0,1}, \ldots, x_{0,N}\}$, *then* m_0 *is the unique solution to* $\mathcal{P}_0^M(y_0)$, *and there exists* $\gamma > 0$, $\lambda_0 > 0$ *such that for* $0 \leqslant \lambda \leqslant \lambda_0$ *and* $\|w\|_2 \leqslant \gamma\lambda$,

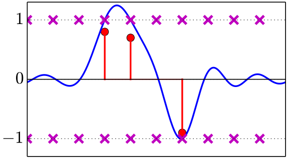
- *The solution* m_λ *to* $\mathcal{P}_\lambda^M(y_0 + w)$ *is unique.*
- $\operatorname{Supp} m_\lambda = \operatorname{Supp} m_0$, *that is* $m_\lambda = \sum_{i=1}^{N} a_{\lambda,i}^M \delta_{x_{\mathbf{0},i}}$, *and* $\operatorname{sign}(a_{\lambda,i}) = \operatorname{sign}(a_{0,i})$,
- $a_{\lambda,I}^M = a_{0,I} + \Phi_{x_{\mathbf{0}}}^+ w - \lambda(\Phi_{x_{\mathbf{0}}}^* \Phi_{x_{\mathbf{0}}})^{-1} \operatorname{sign}(a_{0,I})$.

If $|\eta_F(\frac{k}{M})| > 1$ *for some* k, *the support is not stable.*

Fuchs theorem

For $m_0 = \sum_{i=1}^{M} a_{0,i}\delta_{x_{\mathbf{0},i}}$, define

$$\eta_F = \Phi^* p_F, \text{ where}$$
$$p_F = \operatorname{argmin}\{\|p\|_{L^2(\mathbb{T})}; (\Phi^* p)(x_{0,i}) = \operatorname{sign}(a_{0,i})\}$$
$$= \Phi_{x_{\mathbf{0}}}^{+,*} s.$$



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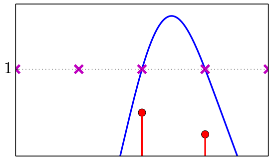
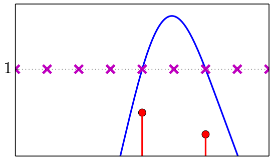
- $a_{\lambda,I}^M = a_{0,I} + \Phi_{x_{\mathbf{0}}}^+ w - \lambda(\Phi_{x_{\mathbf{0}}}^* \Phi_{x_{\mathbf{0}}})^{-1} \operatorname{sign}(a_{0,I})$.

If $|\eta_F(\frac{k}{M})| > 1$ *for some* k, *the support is not stable.*

But. . .





When the grid is too thin, the Fuchs criterion cannot hold
$\Rightarrow$ the support is *not stable*.

Question

What is the support at low noise when the Fuchs criterion does not hold?

The solution at low noise is supported on its **extended support** [Dossal (07)].

The extended support

- The solution to the Lasso $\mathcal{P}_\lambda^M(y_0 + w)$ is **piecewise affine** in (a_0, w, λ) [Osborne et al. (00)].
- Except for a Lebesgue negligible set of a_0, the last path in (w, λ) is of the form:

$$a_\lambda^M = a_0 + \Psi w + \lambda \beta.$$

  for some linear operator $\Psi : L^2(\mathbb{T}) \to \mathbb{R}^M$ and some $\beta \in \mathbb{R}^M$.
- The extended support is determined by $\operatorname{Im} \Psi + \operatorname{Span} \beta$.

Goal

Identify the extended support for the deconvolution problem.

Extended support on thin grids

- Consider a sequence of refining grids with vanishing stepsize:

$$\mathcal{G}_n = \left\{ \frac{i}{M_n} ;\ 0 \leqslant i \leqslant M_n - 1 \right\} \subset \mathbb{T} \quad \text{with} \left\{ \begin{array}{l} \mathcal{G}_n \subset \mathcal{G}_{n+1} \text{ (e.g. } M_n = \frac{1}{2^n}), \\ \lim_{n\to+\infty} M_n = +\infty, \end{array} \right.$$

- Assume that $\operatorname{Supp} m \subset \mathcal{G}_n$ for n large enough, *i.e.*

$$m = \sum_{i=1}^{N} \alpha_{0,i} \delta_{x_{0,i}} = \sum_{k=0}^{M_n-1} a_{0,k} \delta_{\frac{k}{M_n}}.$$

Theorem (D.-Peyré (13,15))

If m_0 is "non-degenerate", for n large enough, the extended support of m_0 on $\mathcal{G}_n$ is given by

$$I \cup \{ i + \varepsilon_i \ ;\ i \in I \} \quad \textit{where} \quad I = \{ i \in [\![0, M_n - 1]\!] \ ;\ a_{0,i} \neq 0 \} \ \textit{and}\ \varepsilon \in \{\pm 1\}^{|I|}.$$

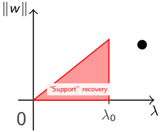
Moreover, ε does not depend on n, and is given by

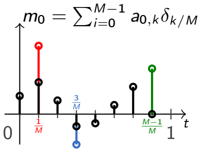
$$\varepsilon = (\operatorname{diag}(\operatorname{sign}(\alpha_0))) \operatorname{sign} \left(({\Phi'_{x_0}}^* \Pi \Phi'_{x_0})^{-1} {\Phi'_{x_0}}^* \Phi_{x_0}^{+,*} \operatorname{sign}(\alpha_0) \right).$$

where Π is the orthogonal projector onto $(\operatorname{Im} \Phi_x)^\perp$

Low noise "robustness" on thin grids

Under the same hypotheses:





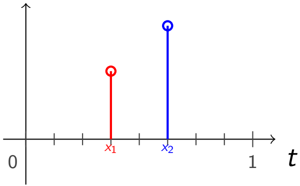
Theorem (D.-Peyré (15))

There exists $\gamma^{(n)} > 0$, $\lambda_0^{(n)} > 0$ such that for $0 \leqslant \lambda \leqslant \lambda_0^{(n)}$ and $\|w\|_2 \leqslant \gamma^{(n)}\lambda$,

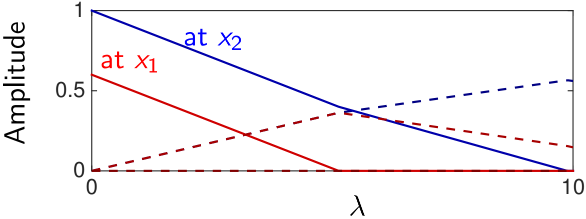
- *The solution $m_\lambda^{(n)}$ to $\mathcal{P}_\lambda^{M_n}(y_0 + w)$ is unique.*
- $\operatorname{Supp} m_\lambda^{(n)} = \bigcup_{1\leqslant i\leqslant N} \left\{x_{0,i}, x_{0,i} + \frac{\varepsilon_i}{M_n}\right\}$, *that is* $m_\lambda^n = \sum_{i=1}^N \left(a_{\lambda,i}^{(n)}\delta_{x_{0,i}} + b_{\lambda,i}^{(n)}\delta_{x_{0,i}+\frac{\varepsilon_i}{M_n}}\right)$, *and* $\operatorname{sign}(a_{\lambda,i}) = \operatorname{sign}(b_{\lambda,i}) = \operatorname{sign}(a_{0,i})$,
- $\begin{pmatrix} a_{\lambda,I}^{(n)} \\ b_{\lambda,I}^{(n)} \end{pmatrix} = \begin{pmatrix} a_{0,I} \\ 0 \end{pmatrix} + \Phi_{x_0,x_0+\varepsilon}^{+} w - \lambda(\Phi_{x_0,x_0+\varepsilon}^{*}\Phi_{x_0,x_0+\varepsilon})^{-1}\begin{pmatrix} \operatorname{sign}(a_{0,I}) \\ \operatorname{sign}(a_{0,I}) \end{pmatrix}.$

In fact $\gamma^{(n)} = O(1)$ and $\lambda_0^{(n)} = O(\frac{1}{M_n})$.

Numerical experiments

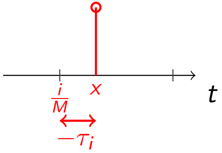


Input Signal



Variant: the "Continuous" Basis Pursuit

A semi-discrete approach :
try to "interpolate" the positions.



- **Idea:** if x is not on the grid:

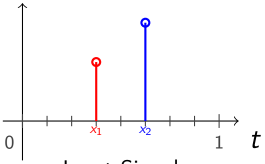
$$
\begin{aligned}
a\varphi(\cdot - x) &\approx a\varphi(\cdot - \frac{i}{M}) + a\varphi'(\cdot - \frac{i}{M})(-x + \frac{i}{M}) \\
&= a\varphi(\cdot - \frac{i}{M}) + a\tau_i\varphi'(\cdot - \frac{i}{M}) \quad \text{where} \quad \tau_i = -x + \frac{i}{M}.
\end{aligned}
$$

- Set $b_i = 2a\tau_i M$, so that $|b_i| \leqslant a$ is $a \geqslant 0$.
- Solve the Continuous Basis Pursuit [Ekhanadam (11)]

$$
\inf_{\substack{(a,b)\in\mathbb{R}^+\times\mathbb{R} \\ |b|\leqslant a}} \lambda\|a\|_{\ell^1} + \frac{1}{2}\|\Phi a + \frac{1}{2M}\Phi' b - (y_0 + w)\|_2^2
$$

Study on thin grids

- The Continuous Basis Pursuit is equivalent to a Lasso with positivity constraint.
- Extended support: again pairs of spikes.
- Stability constants: $\lambda_0^{(n)} = O(\frac{1}{M_n^3})$



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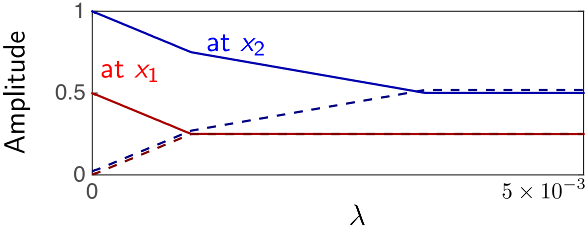
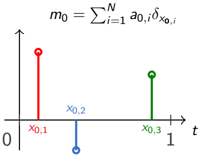


Summary

The total variation

Define the **total variation of the measure** $m \in \mathcal{M}(\mathbb{T})$ as:

$$|m|(\mathbb{T}) = \sup\left\{\int_{\mathbb{T}} \psi dm; \psi \in C(\mathbb{T}), \|\psi\|_\infty \leqslant 1\right\}$$



Example :

- If $m = \sum_{i=0}^N a_i \delta_{x_i}$, then $|m|(\mathbb{T}) = \sum_{i=0}^N |a_i|$.
- If $m = fd\mathcal{L}$, then $|m|(\mathbb{T}) = \int_{\mathbb{T}} |f(t)|dt$.

Such a regularization has been considered in [de Castro & Gamboa (12), Candes & Fernandez-Granda (13), Bredies & Pikkarainen (13), Recht et al. (12)].

Continuous framework for deconvolution

Using the **total variation of measures**:
$|m|(\mathbb{T}) = \sup \left\{ \int_{\mathbb{T}} \psi dm; \psi \in C(\mathbb{T}), \|\psi\|_\infty \leqslant 1 \right\}$

- **Basis Pursuit for measures** [de Castro & Gamboa (12), Candes & Fernandez-Granda (13)],

$$\inf_{m \in \mathcal{M}(\mathbb{T})} |m|(\mathbb{T}) \text{ such that } \Phi m = y_0 \qquad (\mathcal{P}_0^\infty(y_0))$$

- **LASSO for measures** [Recht et al. (12), Bredies & Pikkarainen (13), Azais et al. (13)]

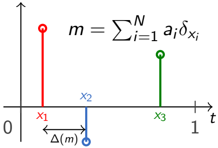
$$\inf_{m \in \mathcal{M}(\mathbb{T})} \lambda |m|(\mathbb{T}) + \frac{1}{2} \|\Phi m - (y_0 + w)\|_2^2 \qquad (\mathcal{P}_\lambda^\infty(y_0 + w))$$

Numerical methods for solving $\mathcal{P}_0(y_0)$ and $\mathcal{P}_\lambda(y_0 + w)$ are proposed in [Bredies & Pikkarainen (13), Candes & Fernandez-Granda (13)]

Identifiability for discrete measures

Minimum separation distance of a measure m:

$$\Delta(m) = \min_{x,x' \in \mathrm{Supp}\, m, x \neq x'} |x - x'|$$



Ideal Low Pass filter: $\varphi(t) = \frac{\sin(2f_c+1)\pi t)}{\sin \pi t}$
i.e $\hat{\varphi}_n = 1$ for $|n| \leqslant f_c$, 0 otherwise.

Theorem (Candès & Fernandez-Granda (2013))

Let φ be the ideal low-pass filter. There exists a constant $C > 0$ such that, for any (discrete) measure m_0 with $\Delta(m_0) \geqslant \frac{C}{f_c}$, m_0 is the unique solution of

$$\inf_{m \in \mathcal{M}(\mathbb{T})} |m|(\mathbb{T}) \text{ such that } \Phi m = y_0 \qquad (\mathcal{P}_0(y_0))$$

where $y_0 = \Phi m_0$.

Remark: $1 \leqslant C \leqslant 1.87$.

Limit of the functionals

We say that $m_n \in \mathcal{M}(\mathbb{T})$ weakly * converges towards $m \in \mathcal{M}(\mathbb{T})$ if

$$\forall f \in C(\mathbb{T}), \lim_{n \to +\infty} \int_{\mathbb{T}} f \, \mathrm{d} m_n = \int_{\mathbb{T}} f \, \mathrm{d} m.$$

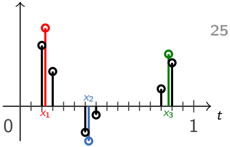
Consider a sequence $(m_n)_{n \in \mathbb{N}} \in \mathcal{M}(\mathbb{T})^{\mathbb{N}}$ such that each m_n is a minimizer of $\mathcal{P}_0^{M_n}(y_0)$ (resp. $\mathcal{P}_\lambda^{M_n}(y_0 + w)$).

Theorem ([Tang et al. 13])

*The sequence $(m_n)_{n \in \mathbb{N}}$ has convergent subsequences for the weak * convergence, and each limit point is a minimizer of $\mathcal{P}_0^{\infty}(y_0)$ (resp. $\mathcal{P}_\lambda^{\infty}(y_0 + w)$).*

Remark: In fact $\mathcal{P}_0^{M_n}(y_0)$ (resp. $\mathcal{P}_\lambda^{\infty}(y_0 + w)$ Γ-converges towards $\mathcal{P}_0^{\infty}(y_0)$ (resp. $\mathcal{P}_\lambda^{\infty}(y_0 + w)$).

Fine properties of the support



More precisely, if the solution $m^\infty = \sum_{i=1}^N \alpha_i \delta_{x_i}$ to $\mathcal{P}_\lambda^\infty(y_0 + w)$ (resp. $\mathcal{P}_0^\infty(y_0)$) is "non-degenerate",

- then the solution m_n to $\mathcal{P}_\lambda^\infty(y_0 + w)$ (resp. $\mathcal{P}_0(y_0)$) is made of pairs consecutive spikes:

$$m_n = \sum_{i=1}^N (a_i \delta_{k_i/M} + b_i \delta_{(k_i+\varepsilon_i)/M})$$

$$\text{with } \operatorname{sign}(a_i) = \operatorname{sign}(b_i) = \operatorname{sign}(\alpha_i), \varepsilon_i \in \{\pm 1\}$$

- At low noise, if the original measure is on the grid, pairs of consecutive spikes (including the original one) (see Section 1) .

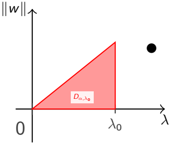
Outline of the argument

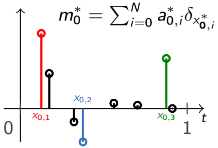
- Approximate the solution of the **dual** to $\mathcal{P}_\lambda^M$ with the solution of the dual to $\mathcal{P}_\lambda^\infty$,
- Control the properties (saturations at ± 1) of that solution
- Deduce the set where Dirac masses may appear in the primal problem.

Robustness of the support (continuous problem)

For $m_0 = \sum_{i=1}^{N} a_{i_0} \delta_{x_{0,i}}$, define

$$\Gamma_{x_0} = \left(\varphi(\cdot - x_{0,1}), \ldots \varphi(\cdot - x_{0,N}), \varphi'(\cdot - x_{0,1}), \ldots \varphi'(\cdot - x_{0,N}) \right)$$





Theorem (D.-Peyré 2013)

Assume that Γ_{x_0} has full rank, and that m_0 satisfies the ***Non Degenerate Source Condition***.
Then there exists, $\alpha > 0$, $\lambda_0 > 0$ such that for $0 \leqslant \lambda \leqslant \lambda_0$ and $\|w\|_2 \leqslant \alpha\lambda$,

- *the solution m_λ to $\mathcal{P}_\lambda(y + w)$ is unique and has exactly N spikes, $m_\lambda = \sum_{i=1}^{N} a_{\lambda,i} \delta_{x_{\lambda,i}}$,*
- *the mapping $(\lambda, w) \mapsto (a_\lambda, x_\lambda)$ is C^1.*
- *the solution has the Taylor expansion*

$$\begin{pmatrix} a_\lambda \\ x_\lambda \end{pmatrix} = \begin{pmatrix} a_0 \\ x_0 \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & \operatorname{diag} a_0^{-1} \end{pmatrix} (\Gamma_{x_0}^* \Gamma_{x_0})^{-1} \left[\begin{pmatrix} \operatorname{sign}(a_0) \\ 0 \end{pmatrix} \lambda - \Gamma_{x_0}^* w \right] + o \begin{pmatrix} \lambda \\ w \end{pmatrix}$$

Conclusion

- (Almost)-stability of the support for the deconvolution problem
- As the grid stepsize refines, stability decreases
- Try the grid free approaches the Sparse Spikes Deconvolution on Numerical tours!

`www.numerical-tours.com`

Papers:
Exact Support Recovery for Sparse Spikes Deconvolution,
V. Duval & G. Peyré (JFoCM 2014)
Sparse Spikes Deconvolution on thin Grids
V. Duval & G. Peyré (ArXiv Preprint 2015)

Thank you for your attention!

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Algorithm (Candès & Fernandez-Granda)

How to solve $\mathcal{P}_0(y)$ in the case of the ideal LPF? $\longrightarrow$ use the **Fourier coefficients**.

- Solve

$$\sup_{p \in L^2(\mathbb{T})} \langle y, p \rangle \quad \text{s.t.} \quad \sup_{t \in \mathbb{T}} |(\Phi^* p)(t)| \leqslant 1.$$

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- Solve

$$\sup_{c \in \mathbb{R}^{2f_c+1}} \Re\langle \hat{y}, c\rangle \quad \text{s.t.} \quad \sup_{t\in\mathbb{T}} \left| \sum_{n=-f_c}^{f_c} c_n e^{2i\pi n t} \right| \leqslant 1.$$

Lemma (Dumitrescu)

A causal trigonometric polynomial $\sum_{n=0}^{M-1} c_n e^{2i\pi n t}$ is bounded by one in magnitude if and only if there exists a Hermitian matrix $Q \in \mathbb{C}^{M\times M}$ such that

$$\begin{bmatrix} Q & c \\ c^* & 1 \end{bmatrix} \succeq 0 \text{ and } \sum_{i=1}^{M-j} Q_{i,i+j} = \begin{cases} 1, j = 0 \\ 0, j = 1, 2 \ldots M-1 \end{cases}$$

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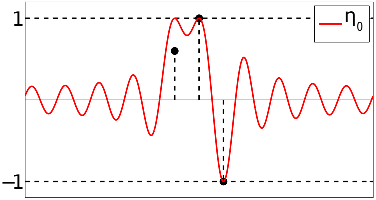
There is a variant for $\mathcal{P}_\lambda(y)$ (Azais et al., 2013)

Example

EXAMPLE

The Non Degenerate Source Condition



Definition

A measure $m_0 = \sum_{i=1}^{N} a_{0,i}\delta_{x_{\mathbf{0},i}}$ satisfies the **Non Degenerate Source Condition** if

- There exists $\eta \in \operatorname{Im}\Phi^*$ such that $\eta \in \partial|m_0|(\mathbb{T})$, or equivalently:
  - there exists a solution p to $\mathcal{D}_0(y)$,
  - m_0 is a solution to $\mathcal{P}_0(y)$
- The minimal norm certificate $\eta_0 = \Phi^* p_0$ satisfies
  - For all $s \in \mathbb{T} \setminus \{x_{0,1}, \dots x_{0,N}\}$, $|\eta_0(s)| < 1$,
  - For all $i \in \{1, \dots N\}$, $\eta_0''(x_{0,i}) \neq 0$.